RESEARCH ARTICLE

European Journal of Education and Pedagogy
www.ej-edu.org

Utilization of "Byrne's Euclid" in the Teaching of Geometry to Students with Special Learning Difficulties: A Qualitative Research

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ABSTRACT

The choice of the appropriate instructional method for teaching Euclidean Geometry to children with Special Learning Difficulties is an important topic. In this paper we study some theoretical issues related to the teaching of geometric concepts to students with Special Learning Difficulties, focusing on the teaching material. We present the design, implementation and results of a qualitative research conducted with six 9th grade students in a Special Vocational High School in Greece, on the use of the website "Byrne's Euclid" in the teaching of propositions of Euclidean Geometry. The research showed that the children understood the key points of the proof of the proposition we presented and they assimilated basic geometric concepts and processes using colors and figures. The paper concludes with a discussion upon suggestions, perspectives and limitations for teaching that utilizes Byrne's Euclid.

Keywords: Byrne's Euclid, Euclidean Geometry, history of mathematics, ICT in mathematics education, special learning difficulties.

Published Online: March 29, 2023

ISSN: 2736-4534

DOI: 10.24018/ejedu.2023.4.2.623

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I. Introduction

The term "Specific Learning Difficulties" (SLD) refers to a special category of difficulties related to learning and mainly to the acquisition of written word (Willcutt *et al.*, 2019). Mathematics as a structure and mathematical concepts in general are considered very important both for the academic progress of people with SLD and for independent living (Barnes *et al.*, 2020). A basic condition for teaching mathematics to children with SLD is the adaptation of materials to their interests and the use of various alternative media, such as visual objects, the use of colors and innovative technologies (Jitendra *et al.*, 2018). One of the recommended strategies for teaching mathematics to children with SLD is the use of visual materials, tools and aids, as well as the multisensory method in the educational process (Marita & Hord, 2017). Research has highlighted the positive learning results of using visual methods in teaching children with SLD (Grobecker & De Lisi, 2000; Cass *et al.*, 2003).

In this article we focus on teaching geometry to children with SLD, using the innovative website "Byrne's Euclid". The English civil engineer Oliver Byrne (1810–1880) wrote, among many other interesting works on mathematics and engineering, *The first six books of Euclid*, published in London in 1847 (Hawes & Kolpas, 2015). In this book, instead of using words and algebraic equations to find angles, sides and figures, color representations are used. In other words, in Byrne's Euclid, words have been replaced by colours and symbols. "Byrne's Euclid" was transferred to electronic version (see here) in 2018 by Nicholas Rougeux, a web designer, and was quickly translated into Greek and other European languages.

Given that Euclidean Geometry has always been a key part of the curriculum for students with typical and non-typical development in Greek Secondary Education, we implemented a five-hour teaching session in Geometry in a special vocational high school in Greece. The children who participated in our study (4 boys and 2 girls) exhibit SLD and attend the 9th grade. Our teaching intervention focused on the presentation and proof through the electronic version of Byrne's Euclid of Proposition I.5 of Euclid's *Elements*: "In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another". For the implementation of our teaching intervention, we separated the students of the class into two groups. Each group was taught the proof of Proposition I.5 using both Byrne's Euclid and the traditional teaching method.

The results of our research showed that the children, through the use of Byrne's Euclid, understood key points of the proof of Proposition I.5 as well as basic concepts of the equality of triangles, such as a) finding the triangles we want to examine in terms of equality, b) the criterion we use to prove that triangles are equal, and c) the identification of the common elements of the triangles.

The children with SLD are a significant proportion (5–15% depending on the diagnostic criteria) of children attending general and special schools worldwide (Grigorenko *et al.*, 2020). The use of visual applications and media (such as the use of videos, 3D tables, color teaching software, etc.) can help to teach Euclidean Geometry more effectively to these children, taking for granted to their difficulties in reading and understanding written text. Our research contributes to the use of Byrne's Euclid as a visual tool for teaching Euclidean Geometry and to the implementation of visual teaching interventions in the teaching of Euclidean Geometry to children with SLD.

II. Theoretical Framework

A. Literature Review

According to the literature, the use of visual applications and colors in the teaching of Mathematics to children with SLD has positive learning results. Kellems *et al*. (2020) examined the effectiveness of using video-based mathematical instruction using augmented reality exercises with secondary school children with SLD. The results of the research showed that all children improved their learning performance in three of the four problem categories given to them by the researchers after the teaching intervention.

Mammarella *et al*. (2013) assessed 16 children with verbal learning difficulties in seven areas of geometry (symmetry, transformations, etc.). In the Euclidean Geometry area, children obtained a high percentage of correct answers when presented with large-sized shapes, demonstrating that visual-spatial memory is critical for understanding Euclidean Geometry. In a literature review conducted by Soares *et al*. (2018), they study SLD and in particular the reading difficulties faced by children, making reference to neurological and genetic studies, as well as new behavioral and instructional interventions applied to their teaching. Researchers report that the use of technology is effective in teaching mathematics to children with SLD. Another literature review conducted by Chatzivasileiou and Driga (2022), showed that the use of Information and Education Technologies (ICT), the use of mobile phones, the use of computers and educational software in the teaching of mathematics, can improve working memory, visual ability, mathematical abilities and metacognitive skills (such as problem solving) of children with SLD.

Toney *et al*. (2013) studied the use of colors as a strategy for teaching Euclidean Geometry by an undergraduate mathematician teaching Geometry in a Secondary School. The results showed that using colors and techniques similar to those described in Byrne's Euclid to prove the bisector of an angle theorem (each point of the bisector of an angle is equidistant from the sides of the angle) provided equal opportunities for all students, with or without learning difficulties, to participate successfully in Geometry lessons. Cass *et al*. (2003) conducted research on teaching perimeter problems to 13-year-old students with SLD. The researchers used various visual aids based on the use of color and manipulative materials to teach them. The result of the research was that students with SLD were able to understand the types of triangles and distinguish them.

Few surveys have been conducted on the use of Byrne's Euclid in teaching typically developing children. Bolondi and Luiqini (2018) used Byrne's Euclid and the dynamic geometry environment "Geogebra" to prove propositions of Euclid's Elements to 16-year-old students in a secondary school in Italy. Their research showed that after the eight hours of instruction, the students were able to understand the proofs using Byrne's Euclid and produce, through "Geogebra" micro-applications to prove Proposition I.5 of the *Elements*. Pedeferri (2020) mentions the importance of Byrne's Euclid in the teaching of Geometry. More specifically, the use of the diagrams and visual representations present in this work can be a very important tool for teaching Geometry and Mathematics in general. Cirafici (2018) presents key elements from Byrne's Euclid and focuses on its pedagogical approach. In particular, she explores how the power of imagery and the presentation of geometric shapes can help children retain abstract mathematical concepts in their memory.

Considering all the above, in our research we posed the following research question, which focuses on 9th grade students with SLD: Does teaching basic geometric concepts and procedures utilizing Byrne's Euclid have better learning outcomes than traditional instruction?

B. The Teaching of Geometry in Greek High Schools

Geometry is taught in Greece for one hour per week in both General High Schools and in Special Vocational High Schools, on the basis of the same Curriculum. The textbook is unique and includes, in addition to Euclidean Geometry, basic elements of Algebra, which are taught three hours per week in General High School and two hours per week in Special Vocational High School. The content of the course includes basic geometric concepts, propositions and theorems, such as the equality of triangles, the similarity of figures, and basic elements of trigonometry.

Clements and Battista (2013) provide a comprehensive definition of Euclidean Geometry taught in schools, describing it as the learning of objects, relations, and transformations of space that have been formalized, and the mathematical systems that have been created to represent them. This definition fits well with the Greek school reality. Indeed, reasoning about space (especially two-dimensional space) involves cognitive processes whereby the mental representations used to describe spatial objects, relations and transformations are constructed and managed by the students themselves with the guidance of the teacher.

The purpose of the Geometry course is to cultivate children's logical thinking through the understanding of basic relations and theorems of Euclidean Geometry. At the same time, the lesson aims to familiarize students with solving geometry problems and understanding the structure used to prove geometric propositions.

However, the teaching of Geometry in the Special Vocational High Schools, as in the General High Schools, is based on the traditional teaching model. The teacher presents the course material on the blackboard and then the children "mechanically" copy what they see on the blackboard into their notebooks. This method creates many questions for children with SLD, as they find it difficult to understand the written word and do not understand what they copy from the blackboard.

Knowing the usual school practice, we considered that our teaching intervention could capture students' attention and bring about better learning outcomes than traditional teaching. The empirical findings that emerged probably confirm, at least to some extent, our teaching choices. For example, one student who participated in the study stated:

"Now I finally understand how we use the side-angle-side criterion and what it means that two triangles are equal".

C. Elements of the Curriculum and the Textbook

The expected learning outcomes for the Geometry unit, according to the Curriculum of the 9th grade for High Schools, are the following: 1) Students should be able to examine the role of triangle equality criteria in comparing triangles and relate them to the definition of triangle equality; and 2) Students should be able to use triangle equality criteria to justify properties of lines (midpoint of a line segment, bisector of an angle, etc.) and figures (e.g., parallelograms).

In addition, detailed teaching instructions for each unit are given by the Greek Ministry of Education at the beginning of each school year. These instructions, for the equality of triangles unit, focus on reviewing basic geometric knowledge, exposing students to aspects of the mathematical proof process, and highlighting the importance of the equality (congruence) of triangles criteria.

The textbook used for the teaching of 9th grade is unique and is provided free of charge by the Greek Ministry of Education. Part A, entitled "Algebra", includes algebraic expressions, equations, inequalities and the study of basic functions. Part B, entitled "Geometry-Trigonometry", includes equality of triangles, ratios and proportions, similarity of figures and elements of trigonometry, such as basic trigonometric identities.

More specifically, the section on the equality of triangles includes some recurring concepts related to triangles (basic concepts, sides, angles, types of triangles, etc.). In addition, the equality criteria for random and right triangles are given without their proofs, as well as some propositions on the equality of triangles, again without proof. The exercises contained in this module are proof exercises focusing on a good knowledge of the criteria of equality of triangles. The classroom teacher has the option of assigning students additional homework related to the above material.

Especially in recent years the teaching of Geometry in the Greek high school has been limited and has taken on a formalistic character (i.e. is focused on solving techniques). There is no discussion on historical and cultural contexts that influenced the formation of Euclid's *Elements* and the evolution of geometry up to the analytic geometry and the non-Euclidean geometries. At the end of each chapter of the textbook there is a historical note, which however is not part of the examinable material, so it is usually ignored by both educators and students (Rizos & Adam, 2022).

III. Overview of the Project

A. Preliminaries

Already from the academic year 2021–2022 we had the intention to implement qualitative research to investigate whether the use of visual aids in Geometry can help students with SLD to achieve better learning outcomes compared to traditional teaching. Thus, in the first weeks of the academic year 2022–2023, one of the authors of the present study, who teaches Mathematics in a Special Vocational High School, spent 8 teaching hours in order to do a general and in-depth review on triangles (elements and types of triangles, sum of angles, etc.) with his students. It should be noted here that in the Special Vocational High Schools in Greece there is an additional class (10th grade) compared to High Schools, so it is up to the discretion of each teacher to manage his lesson material as he/she sees fit, depending on the requirements and needs of his/her students. Along with the review, we refined the design of our teaching intervention, the implementation of which began a few weeks after the review was completed.

B. The Daily Life of the School Where the Research Was Carried Out

The Special Vocational High School, in which the research was carried out, is a modern school complex. It serves students from all over the region (somewhere in central Greece) and the local Municipality provides buses and taxis for the transport of the children enrolled in it. It consists of 15 spacious classrooms, which accommodate 70 students aged 12 to 18 every day. The classrooms have the classic blackboard and whiteboard, but in recent years they have been equipped with new interactive boards and projectors. The school has assistive barriers for people with mobility problems, in order to facilitate their access to all areas. It has a modern computer laboratory, an electrical engineering laboratory and an agricultural laboratory for students who are following the relevant orientation in their studies.

All educators have specific training, often participate in training activities and upload teaching material to the e-class, while informing students how to study this material on their computers and tablets. The cases of students attending this school are representative of the students attending Special Vocational High Schools across the country. More specifically, the school is attended by students on the autism spectrum, students with SLD, students with mobility problems, students with mild and moderate mental retardation, etc. The school has a friendly atmosphere among all pupils and between pupils and teachers.

C. The Participating Students

The 9th grade class that participated in our research consisted of 4 boys and 2 girls. The students come from the same cultural background and we know them personally from the previous school year. More specifically (we use pseudonyms in order to preserve their anonymity):

Panos is 15 years old and has mastered basic geometric skills. However, he has difficulty recalling prior mathematical knowledge when it is needed to solve problems.

Maria is 15 years old and has mastered basic geometry skills. She has difficulty recognizing geometric shapes and relationships.

Thanos is 16 years old, has a low mathematical background and has difficulty concentrating in class. He has difficulty understanding basic geometric relationships and recalling them in his memory.

Ioanna is 15 years old and her mathematical background is quite low. She knows basic geometric shapes but has difficulty drawing them.

Gregory is 16 years old and has a good mathematical background. He can understand basic geometric relationships and can draw geometric shapes.

Kostas is 15 years old and has a good mathematical background. He can draw shapes and can memorize mathematical relationships.

In addition, everyone has difficulty copying what they see on the board into their notebook. In particular, they have difficulty understanding the mathematical symbols and understanding the content of what they are copying.

IV. Our Teaching Intervention

In December 2022 we conducted a five-hour teaching session in one of the two sections of the 9th grade of a Special Vocational High School on the teaching of Proposition I.5 of Euclid's *Elements*: "In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another", which is part of the "Equality of triangles" section of the textbook, using Byrne's Euclid, and comparing Byrne's Euclid with the traditional teaching method used in school.

In the first hour of the five-hour class, bearing in mind that teaching with references to the History of Mathematics can have a positive effect on students' engagement (Rizos & Gkrekas, 2023) we took a few minutes to talk to children about Euclid's work and life in Alexandria in 300 BC, while showing them some relevant images on the interactive board. Then the electronic version of Byrne's Euclid was briefly presented. In addition, the students were divided into two groups in a random manner. The first group was to be taught Proposition I.5 of the *Elements* using the traditional blackboard teaching method, while the other group was to be taught the same proposition using Byrne's Euclid.

When Byrne's short presentation of Euclid was given on the interactive board, all the students were impressed. In particular, the students were delighted with the use of different colors used to illustrate the sides and angles of triangles. Dividing the groups therefore proved to be a difficult process, as all the students wanted to participate in the group that would prove the proposition using Byrne's Euclid. For this reason, we announced to the students that after the end of the lesson, the groups would be reversed and the learners who had been taught Proposition I.5 of the *Elements* using traditional teaching would then be taught it using Byrne's Euclid, and vice versa for the other group. Thus, the three-hour teaching we had originally planned became five. The groups were therefore divided as follows:

Group A: Panos, Maria and Thanos, who were to be taught Proposition I.5 using Byrne's Euclid.

Group B: Ioanna, Gregory and Kostas, who would teach Proposition I.5 using the traditional teaching method.

In the second hour of our teaching intervention, the teaching of proposition I.5 of the *Elements* to Group A took place. First, we presented the proposition on the interactive board utilizing Byrne's Euclid (see Fig. 1). The children understood the content of the proposition and what we were asked to prove, without expressing any questions. More specifically, Group A students understood the content of the proposition, and which angles we want to prove are equal by the colors, which replace the words and geometric symbols used in propositions in Geometry.

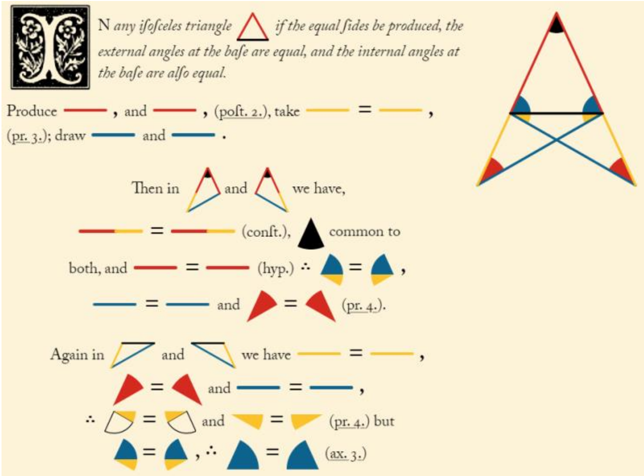

Fig. 1. Proposition I.5 exactly as presented in Byrne's Euclid.

In fact, Maria raised her hand vigorously and with a big smile told us: "Sir, this is the first time I understand directly what I have to prove in Geometry because of the colors that are there, and I don't have to read the exercise we have to deal with several times to understand what we have to do". The rest of the learners in Group A watched the presentation of the proposition with great fascination and total concentration. When we proceeded to prove it, all the students understood which triangles to compare first, using the Side-Angle-Side (SAS) criterion. More specifically, when we moved the computer mouse over the triangles to be compared and the elements that were equal were shown in the same colors, all the students in the group showed great enthusiasm because they fully understood why the triangles we were comparing were equal. In this context, Panos said with joy and a strong feeling of confidence, "Now I finally understand how we use the SAS criterion and what it means that two triangles are equal. I understood the concept of the common angle of two triangles now that you have shown it with the colors".

To complete the proof, we had to compare two more triangles to prove that they are equal and to come to the conclusion. The learners understood the elements that the triangles have so that they are equal with the help of the colors, and using the SAS criterion we were led to the conclusion of the proof of the proposition. When the proof was completed, Thanos, with a long extension of his hand, showing satisfaction, said: "I would like from now on all our Geometry lessons to be done on the interactive Byrne's Euclid board".

The third hour was spent teaching proposition I.5 of the *Elements* to Group B following the traditional method. First, we presented the proposition on the blackboard by copying the pronunciation from the textbook and making a figure using a ruler. As the students copied the proposition in their notebooks, they had a hard time to understand its content and what we were trying to prove. Kostas, with a look of puzzlement, asked: "What are the angles under the base and how do we extend the sides on our own?".

Then we showed the students which triangles to compare, first, to see if they are equal or not. They had difficulty identifying which elements of the triangles we were comparing were equal. Also, the students had difficulty identifying on their own which triangles to compare, as a result of which they are disappointed. In fact, Kostas and Ioanna grabbed their heads, since they could not understand what they were copying from the blackboard. Ioanna in a low voice and with a slight extension of her hand said: “Sir, I have understood nothing. I am having great difficulty reading from the blackboard the triangles we have to compare and finding out which elements are the same”.

In the last comparison, the students were very confused. More specifically, they could not find the triangles they needed to compare, since the figure on the blackboard included the previous triangles we had compared. In addition, they had difficulty finding the evidence they needed to prove that the triangles were equal, and as a result they were discouraged.

When the proof was completed, the learners stared at their watches to see when the bell would ring. We asked them if they had understood the proof of the proposition, and the answers we got were disappointing. Gregory crouched down in front of his desk in a very low voice told us: “Sir, I don’t understand Geometry at all. What we write on the blackboard is difficult and I cannot understand it even though I read it many times at home”.

In the fourth hour of our teaching intervention, the dynamic of the groups was changed, since the students of group B, after having been taught the proposition with the traditional teaching method, persistently asked for it again. The groups now became:

Group A: Ioanna, Gregory and Kostas, who were to be taught Proposition I.5 with Byrne’s Euclid.

Group B: Panos, Maria and Thanos who would teach Proposition I.5 using the traditional method.

Proposition I.5 of *Elements* using Byrne’s Euclid on the interactive board was presented to the new group A, who had been taught this proposition using the traditional teaching method a few days before. Once the students saw the colors instead of the letters and geometric symbols they were excited and this time they understood the content of the proposition and all the stages of its proof, without expressing difficulties or questions. More specifically, the students asked us to teach geometry using Byrne’s Euclid, since they were able to identify directly the triangles to compare and the elements they needed to prove that the triangles are equal to each other. Gregory, now full of joy and confidence, raising his hand very high, said: “Sir, can we use colors in Algebra too? The color makes it easier for us to understand what you are telling us without difficulty”. The rest of the children in the group now had happy faces and had shed the anxiety and frustration they had in proving the proposition using the traditional method of teaching.

In the fifth hour, the students of group B were shown the proof of Proposition I.5 following the traditional teaching method, while they themselves had been taught it with Byrne’s Euclid. The students understood the content of the proposition relatively easily, because they remembered it from the teaching of the same proposition utilizing Byrne’s Euclid a few days ago. Then, as we proved the proposition, there were difficulties during the copying from the blackboard, but the learners remembered to a satisfactory degree the triangles they had to compare and the elements of the triangles they had to find from the proof of the same proposition using Byrne’s Euclid. At the end of the lesson, the students asked us to stop using the blackboard and the classical method of teaching geometry because it is confusing and stressful for them, and they cannot understand what we write on the blackboard.

Indeed, Maria smiling but at the same time with a feeling of worry and anxiety, said: “It’s a good thing that two days ago you showed us this proposition in color on the interactive board, otherwise I wouldn’t have understood anything. The different colors helped me to remember the elements of the triangles that are equal and the triangles we had to compare”. The other members of the group agreed with Maria and concluded that Byrne’s Euclid was very helpful in teaching, since they did not have to read text and copy from the blackboard.

Finally, we had a short discussion (5–7 minutes until the bell rang and the fifth and final hour of our lesson was over) with all the students in the class about the proposition and its proof. All stated that using Byrne’s Euclid to teach Geometry made it easier for them to understand the content of the proposition and its proof, and they would like it to be used from now on in all their Geometry lessons going forward. In addition, they asked if there is a similar visual aid in Algebra in order to use it there as well.

The research we implemented in the 9th grade was qualitative and was aimed to study in depth the effects of teaching children with SLD using Byrne’s Euclid. Therefore, the methodology that was considered appropriate in order to answer the research question we posed was the one based on open-ended questions (included in a questionnaire we constructed) and answered by the learners, in order to bring to the surface what they think and why.

V. Analysis of the Research Data

The basic data of our research (apart from the observations we made during the lessons) were drawn from a questionnaire which the students were asked to answer orally, due to the difficulties they face in writing. The questionnaire consisted of three open-ended questions, which each student was asked to answer on their own in the presence of the teacher in the classroom. The three questions were as follows:

Q1. Which parts of Byrne’s Euclid made a particular impression (positive or negative) on you during the teaching? (see in Table I).

Q2. Did you understand the concept of equality of triangles and the SAS criterion of equality of triangles contained in the proof of the Proposition you were taught more easily through the teaching based on “Byrne’s Euclid” or through the traditional teaching method (i.e. the one we are used to following in the classroom)? (see in Table II).

Q3. Would you prefer Byrne’s Euclid or the traditional teaching method for teaching geometry and why? (See in Table III).

After implementing the semi-structured interviews with the students, we collected the data and proceeded to analyze them. We used the content analysis method (Abreu *et al.*, 2022) to analyze the data. More specifically, we proceeded to an initial coding of the data obtained from the students’ interviews.

TABLE I: Q1-WHICH PARTS OF BYRNE'S EUCLID MADE A PARTICULAR IMPRESSION ON YOU DURING THE LESSON?

Interview extracts	Codes
Kostas: The use of colors that I saw on the sides and vertices of the triangles on the interactive board made a positive impression on me. The colors helped me understand which triangles to compare, the equal sides and angles. [...] The words and pronunciations always confused me in Math and I never knew what I was supposed to do. Now I understand what the proof is asking me.	Understanding geometric concepts using colors through Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols
Thanos: I finally understood what I have to do. Oof I got rid of the anxiety I had when we had maths and we were always writing on the blackboard. Now I understand what I have to do in the exercises we do, since the words always confused me. I wish we could use Byrne's Euclid in algebra too. The colored sides and angles helped me understand what it means to extend the sides of a triangle that you explained on the marker board.	Child's negative feeling about mathematics with traditional teaching The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Understanding geometric concepts using colors through Byrne's Euclid
Gregory: Definitely the colors I saw in the figures on the board helped me. When we had Maths and the teacher was writing on the board with a marker, I didn't understand the words and figures. That is, I didn't understand which sides I was supposed to find and which angles. And I was very nervous. Very nervous. When you showed us the proof with Byrne's Euclid, I understood what we needed to find and what we needed to do. I also understood what we meant by side of the triangle and angle of the triangle, since with the colors everything was clear.	Understanding geometric concepts using colors through Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Child's negative feeling about mathematics with traditional teaching Understanding of basic geometric concepts
Maria: First of all, there are no formulas and letters on the table. I always had a hard time copying these things from the blackboard. I really like the interactive board because everything is easier to see. Math is a subject that interests me and the colors I saw on the proof were very helpful.	The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Positive impact of ICT in teaching children with SLD Understanding geometric concepts using colors through Byrne's Euclid
Ioanna: Mathematics is a subject that is very difficult for me. Especially in Geometry, I could not understand what the difference between an angle and a side is and the notations the teacher uses on the board. But once she showed us the triangles with colors and no letters, everything became easier. I understood the side-angle difference and what side extension means. Many times the teacher had written this word on the board but I never understood what it really meant. I want to keep using Byrne's Euclid and the interactive whiteboard because it will make geometry easier for me to understand. I wish there was such an aid in algebra as well.	The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Understanding geometric concepts using colors through Byrne's Euclid Understanding of basic geometric concepts The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Positive impact of ICT in teaching children with SLD
Panos: I don't find anything negative in Byrne's Euclid that we used in the lesson. The colors used to represent angles and sides are very helpful in understanding which triangles to compare and which of their elements are equal. I would also like us to use a similar method in algebra to understand the mathematical relationships we are making there as well. I get very nervous about the letters and symbols we use in both Algebra and Geometry. It makes me very nervous about this process.	Understanding geometric concepts using colors through Byrne's Euclid Positive impact of ICT in teaching children with SLD The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Child's negative feeling about mathematics with traditional teaching

TABLE II: Q2-DID YOU UNDERSTAND THE CONCEPT OF EQUALITY OF TRIANGLES AND THE CRITERION OF EQUALITY OF TRIANGLES "SIDE-ANGLE-SIDE" CONTAINED IN THE PROOF OF THE PROPOSITION YOU WERE TAUGHT MORE EASILY THROUGH THE TEACHING BASED ON "BYRNE'S EUCLID" OR THROUGH THE TRADITIONAL TEACHING METHOD (I.E. THE ONE WE USUALLY FOLLOW IN THE CLASSROOM)?

Interview extracts	Codes
Kostas: Certainly with Byrne's Euclid [...] Previously I didn't understand the SAS criterion. I didn't understand what a contained angle meant and how I could find it in the figure. The colors in Byrne's Euclid helped me to understand how I can identify the elements of the triangles I want to compare, and also the meaning of the included angle. I used to be nervous because I couldn't understand them from the table. I believe geometry should be done with Byrne's Euclid and using an interactive board. The pictures and colors help me a lot to understand.	The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Understanding geometric concepts using colors through Byrne's Euclid Child's negative feeling about mathematics with traditional teaching Positive impact of ICT in teaching children with SLD
Thanos: With Byrne's Euclid [...] Before you showed us the proof with Byrne's Euclid, I couldn't understand the basics of geometry, such as what it means to extend a side [...]	Understanding of basic geometric concepts
Gregory: I didn't understand the SAS criterion very well in the blackboard. This was something that made me sad because I think I'm good at Geometry [...] I had a hard time with the symbolism of triangles and angles because they are similar to each other. [...] The interactive board should be used to teach both Geometry and Algebra.	Child's negative feeling about mathematics with traditional teaching The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Positive impact of ICT in teaching children with SLD
Maria: I'm very happy to tell you that what helped me understand was Byrne's Euclid. I first understood which triangles I had to compare, because the colors pointed them out to me, and I also understood the elements that have equal elements. [...] When I don't understand what I should do, I get anxious and upset.	Feelings of joy and confidence from using Byrne's Euclid Understanding geometric concepts using colors through Byrne's Euclid Child's negative feeling about mathematics with traditional teaching

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TABLE II: CONTINUATION

Ioanna: When we did exercises and proofs on the board [with the traditional teaching method], the letters and relationships confused me and I couldn't understand their meaning. Byrne's Euclid helped me to immediately understand what I was seeing on the interactive board. The colors helped me figure out which triangles to compare and which elements are equal. So this way is much more effective than what we used to do. Now I'm enjoying my math time.	The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Positive impact of ICT in teaching children with SLD Understanding geometric concepts using colors through Byrne's Euclid Feelings of joy and confidence from using Byrne's Euclid
Panos: Of course [the best method is] Byrne's Euclid. Without a second thought. First of all, the colors helped me to understand which triangles to compare and also to understand what the SAS criterion means [...] Copying from the blackboard made it difficult for me, but now it is not necessary.	Feelings of joy and confidence from using Byrne's Euclid Understanding geometric concepts using colors through Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols

TABLE III: Q3-WOULD YOU PREFER BYRNE'S EUCLID OR THE TRADITIONAL METHOD OF TEACHING GEOMETRY AND WHY?

Interview extracts	Codes
Kostas: I would definitely choose Byrne's Euclid for teaching Geometry. The reason is that the colors of the proof helped me to understand it without much difficulty. Also, Byrne's Euclid does not have words and mathematical symbols that often confuse me.	Understanding geometric concepts using colors through Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols
Thanos: [I would choose] Byrne's Euclid. Byrne's Euclid helped me understand and solve basic questions I had in Geometry, such as what a contained angle means and general questions I had about triangles. […] The colors helped me find the triangles to compare without getting confused.	Understanding of basic geometric concepts Understanding geometric concepts using colors through Byrne's Euclid
Gregory: I'm excited about Byrne's Euclid. I would like all lessons from now on in Geometry to be like this. [...] There are many positive points… We don't write symbols and words on the board. These make it difficult for me. [...] I wish there was something similar in Algebra.	Feelings of joy and confidence from using Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Child's negative feeling about mathematics with traditional teaching
Maria: Of course I prefer Byrne's Euclid! There is no comparison to teaching on the blackboard! Just not writing symbols and relationships on the blackboard is the most important thing for me. [...] The colors I saw on the interactive board helped me understand a lot.	Feelings of joy and confidence from using Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Understanding geometric concepts using colors through Byrne's Euclid
Ioanna: Byrne's Euclid with a thousand! Without a second thought! The colors we demonstrated the sentence with helped me understand it straight away. I didn't have to copy anything from the table. That's the hard [...]	Feelings of joy and confidence from using Byrne's Euclid Understanding geometric concepts using colors through Byrne's Euclid The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols
Panos: Obviously I prefer Byrne's Euclid! The letters [variables] and equations we usually use in Mathematics make it quite difficult for me. But now I understand what I need to prove and how to do it [...] I also like using the interactive board […]	The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols Positive impact of ICT in teaching children with SLD

We chose colors instead of codes or letters to represent the themes that emerged from the children's interviews (see in Table IV). The reason we did this was to be closer to the philosophy of Byrne's Euclid, since the use of colors is the key feature for symbolizing sides and corners. The Thematic Sections that emerged from the codes are as follows:

TABLE IV: THEMATIC SECTIONS

Codes	Color
Understanding geometric concepts using colors through Byrne's Euclid	Red
Understanding of basic geometric concepts	Purple
The use of written word causes difficulties for children with SLD in understanding pronunciations and understanding math symbols	Green
Child's negative feeling about mathematics with traditional teaching	Blue
Feelings of joy and confidence from using Byrne's Euclid	Grey
Positive impact of ICT in teaching children with SLD	Brown

The above thematic sections emerged from the analysis of the data collected from the student interviews, and largely indicate that teaching children with SLD using Byrne's Euclid can be more effective than the traditional teaching method.

More specifically, in order to answer our main research question, that is whether Byrne's Euclid can be an effective teaching method, we implemented the following teaching procedure: after the Christmas holidays, in January 2023 (our initial five-hour teaching intervention took place in the last two weeks before school closed in December 2022), we gave students a homework exercise that was similar to the proof of Proposition I.5, and asked them to solve it by any method they wish, either traditional or Byrne's Euclid. In addition, we uploaded the link to the "Byrnes Euclid" website to the e-class, in order for the children to study and recall this method. The next day we found that all the students had solved the exercise using Byrne's Euclid's method.

We then asked each student to solve the same exercise using the classical method on the whiteboard. For this reason, we asked the music teacher and the gymnastics teacher to keep the other students in the class busy while we examined each student separately from the others. The examination took place in the classroom. More specifically:

Panos remembered that we have to produce further the two sides of the triangle and he did it with a different colored marker, showed us the triangles to compare, and with his hand he showed us their elements that are equal, and again with his hand he indicated the next pair of triangles to compare. When Panos was part of the group that learned the proposition we demonstrated in our teaching intervention using the classical teaching method, he had difficulty understanding why we were extending the sides of the triangle and identifying the pairs of triangles he had to compare. As he told us: “Byrne’s Euclid helped me to remember which sides to extend, to identify the triangles to compare”. He showed progress after recalling information from the proof he had done at home with Byrne’s Euclid and using it in the traditional teaching method.

Maria: When she was asked to solve the exercise based on the traditional teaching method, the results were positive. More specifically, she managed to draw the extensions of the lines of the triangle with different colored markers and identify the triangles that needed to be compared. She had a little difficulty in locating the elements of the equal triangles, but eventually located them, as well as locating the next pair of equal triangles. When Maria was in the group that was taught Proposition I.5 with the classical teaching method, she had difficulty understanding the figure we made on the board. She also could not understand the pairs of triangles that needed to be compared.

Thanos was able to understand that we need to produce further the sides of the triangle and he identified on his own the pair of triangles to be compared. However, he could not find all the elements of the triangles required to prove that they are equal. When Thanos was part of the group taught with the classical methodology, he could not understand why we were extending the sides of the triangles. He also did not understand why we were comparing the specific pairs of triangles and why we had to come to the conclusion we were being asked. Thanos stated: “The colors you used on the interactive board and the figures that Byrne’s Euclid had, helped me to understand equal triangles. The colors helped me to understand the equal sides of triangles and how I can identify the triangles to compare”.

Ioanna drew with colored markers the triangle on the whiteboard herself. In addition, she extended the sides of the triangle herself and figured out the original pair of triangles to compare. When she was part of the group that was taught the proof of the proposition using the traditional teaching method, she had difficulty drawing the triangle. She also could neither locate on the board the pairs of triangles that needed to be compared, nor could she extend the sides of the triangle to form the other pair of triangles. Ioanna stated: “The colors and figures in Byrne’s Euclid helped me to understand how to draw the triangle in the homework exercise you gave us and to remember which triangles I needed to compare”.

The results for *Gregory* were encouraging. He was able to draw the triangle given in the exercise (with markers of different colors), extend its sides and identify the pairs of triangles he had to compare each time. The elements of the triangles were described in terms of colors rather than using geometric symbols and relationships. When Gregory was part of the group that was taught Proposition I.5 using the traditional teaching method, he could not understand why we had to extend the sides of the triangle and could not identify the second pair of triangles to compare. Gregory told us: “The proof we did using Byrne’s Euclid helped me to identify directly the pairs of triangles to compare, as well as the elements of the triangles that are equal each time. That’s why I asked for colored markers so I could capture what I’m thinking about in color. The letters make it difficult for me”.

Kostas was able to draw the triangle (also using markers of different colors), identify with relative ease the first pair of triangles he had to compare, extend the sides of the triangle and identify the second pair of triangles he had to compare. When Kostas was in the group that was taught Proposition I.5 using the classical teaching method, he could not understand what we had to prove, since he had difficulty understanding the geometric symbols we used to present the proof. In addition, he could not understand why we were extending the sides of the triangle and was unable to locate the second pair of triangles to compare.

VI. Discussion

The introduction of ICT in the field of Education, already since the 1980s and the widespread interactive educational software (*Cabri Geometry*, *The Geometer’s Sketchpad* etc.), has brought about significant changes in the way Mathematics is taught and learned.

In the last three years, however, especially in Greece, due to the Covid-19 pandemic and the fact that distance education was de facto required, the need for training of teachers on the new digital tools (platforms, courses management systems, collaboration tools etc.) became imperative, while new teaching techniques came to the fore (Rizos & Gkrekas, 2022; Rizos *et al*. 2023). Moreover, by decision of the Greek Ministry of Education, all primary and secondary schools have or will be provided with interactive boards. In this sense, Byrne’s Euclid constitutes a modern visual aid, which seems to be attracting the interest of more and more educators and researchers around the world. However, research on its utilization in children with SLD is very restricted both in Greece and internationally. This paper aims to contribute to filling this gap.

In this article we examine the effect of the visual tool “Byrne’s Euclid” on the teaching of Euclidean geometry to students with SLD in a Special Vocational High School in Greece. Our research revealed the following advantages of using Byrne’s Euclid in the above context:

1) Assimilation of basic geometric concepts,
2) Increase of learning engagement in solving geometric problems,
3) Understanding the main points of a geometric proof,
4) Active involvement of all students in the class in solving geometric problems using colors, and
5) Cultivating a positive attitude towards geometry and mathematics in general.

One of the findings from our teaching intervention is that using Byrne’s Euclid improves students’ mathematical abilities, visual perception, and metacognitive skills such as problem solving. Chatzivasileiou & Driga (2022) agree with this. In addition, there was active participation by all students in Geometry lessons when Byrne’s Euclid was utilized to prove Proposition I.5, a fact with which the research converges (see Toney *et al*., 2013). Furthermore, students were able to retain in their memory many elements of the proof of Proposition I.5 with the figures and colors present in Byrne’s Euclid, which is confirmed by Pedeferri’s (2020) research.

However, there are also limitations that emerged during the research process. The main ones are the following:

1) The time it took the students to deal with the homework posted on the e-class platform, and
2) The learning gaps that the students had in basic geometric concepts.

Students attending schools in Greece have many hours of reading at home every day, making it difficult for them to cope with their obligations when they have an extra workload in a lesson. For example, Gregory told us: "In order to catch up with the homework you gave me, I reduced the time I would have had to study for the other lessons". In addition, students were deficient in basic geometry concepts (e.g., separating a side from an angle), since prior to the use of the interactive board, geometry was taught exclusively on a blackboard or a whiteboard. The use of Byrne's Euclid on the interactive board filled some of the gaps that students had.

In terms of students' achievements after being taught using Byrne's Euclid, these are shown by:

1) Solving the exercise on the e-class platform, with the optional use of Byrne's Euclid versus the traditional way used in the classroom until then,
2) The use of techniques from Byrne's Euclid in solving the same exercise on the classroom blackboard, and
3) The discussion held during the research interviews with each student.

From the research we conducted, it is clear that our research question has a positive answer. That is, teaching basic concepts and procedures to high school students with SLD utilizing Byrne's Euclid website has better learning outcomes than traditional teaching. More specifically, after the end of our teaching intervention, all the participating students were able to grasp the main points of the proof we presented to them, to understand the basic geometric concepts required for the proof of proposition I.5 and to identify on the board on their own the pairs and elements of the triangles they had to compare. In addition, students developed a positive attitude towards geometry and mathematics in general, which was not the case when the basic geometric concepts and exercises were taught using the traditional blackboard teaching method.

VII. Conclusion

Based on the results of the teaching intervention we implemented, it appears that teaching Euclidean Geometry to children with SLD utilizing Byrne's Euclid promotes active participation in solving geometric exercises. Furthermore, Byrne's Euclid can be a useful teaching tool for educators to present basic geometric concepts and procedures of Euclidean Geometry in both Special Vocational High School and High Schools, since the colors and figures it contains contribute to students' better understanding of geometric concepts and procedures. There are children with dyslexia in the High Schools who attend Inclusion Classes, who have difficulties in reading and writing. It is possible that the use of Byrne's Euclid in the teaching of geometry to children with dyslexia attending Integration Sections in mainstream schools may bring about positive learning outcomes. Future research could confirm or reject this conjecture.

Acknowledgment

We thank the editor and the anonymous reviewers for their careful reading of our manuscript.

Funding

The authors received no financial support for the research and/or authorship of this article.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

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